CAT 7

Cellular (co)sheaves

Note 0

a) A persistence module gives us the following data:

- for each $i \in \mathbb{N}$, a vector space V_i
- for each $i \leq j$, a linear map $\varphi_i^j : V_i \longrightarrow V_j$ (given by composing $V_i \xrightarrow{f_i} V_{i+1} \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_{j-1}} V_j$)

b) When describing the stability theorem, we had $\mathbb{N}$ replaced by $\mathbb{R}$, with vector spaces $\{V_t \mid t \in \mathbb{R}\}$ and maps $\{\varphi_s^t : V_s \longrightarrow V_t \text{ for } s \leq t\}$. Those satisfied

$$\boxed{\varphi_t^r \circ \varphi_s^t = \varphi_s^r} \qquad \forall\, s \leq t \leq r$$

Def 1

A CATEGORY C consists of a set/class of OBJECTS C_0, and for each pair $x, y \in C_0$ a set $C(x,y)$ of MORPHISMS (usually denoted $f : x \to y$). Across each triple $x, y, z \in C_0$ there is a COMPOSITION

$$\circ = \circ_{x,y,z} : \underset{(g)}{C(y,z)} \times \underset{(f)}{C(x,y)} \longrightarrow \underset{(g \circ f)}{C(x,z)}$$

subject to two laws:

(i) Associativity: $\forall\ (w \xrightarrow{f} x \xrightarrow{g} y \xrightarrow{h} z)$,

$$(h \circ g) \circ f = h \circ (g \circ f)$$

(ii) Identity: $\forall\ x \in C_0,\ \exists\ \mathbb{1}_x \in C(x,x)$ satisfying $f \circ \mathbb{1}_x = f$ and $\mathbb{1}_x \circ g = g$ $\forall f, g \ldots$

$C(x,x)$ = endo-morphisms of x

Invertible $f : x \to y$ is an isomorphism

etc

Eg 2

(you have seen a lot of these...)

✓ $\underline{SC}$ = "simplicial complexes, simplicial maps"

✓ $\underline{\mathrm{VECT}}_{\mathbb{F}}$ = "$\mathbb{F}$-vector spaces, linear maps"

✓ Every poset, P [$\exists!\ x \to y$ iff $x \leq y$] ← (Partially ordered set)

✓ Every group, G [$\exists!$ object, every $g \in G$ is an invertible endo-morphism]

✗ "Vector spaces, non-invertible linear maps" ← (even fails the identity law!)

Def 3 A FUNCTOR between categories C and D, written $F : C \to D$ assigns

- to each object $c \in C_0$ an object $Fc \in D_0$
- to each $f : c \to c'$ in C a morphism $Ff : Fc \to Fc'$ in D

so that

- $F(\mathbb{1}_c) = \mathbb{1}_{F(c)}$, and
- $F(g \circ f) = F(g) \circ F(f)$

Eg 4 a) Homology is a functor $SC \xrightarrow{H_\bullet} \mathrm{VECT}_{\mathbb{F}}$ in fact, $CH(\mathbb{F})$: "chain complexes, chain maps" is also a category, and the entire pipeline

$$\boxed{SC \xrightarrow{C_\bullet} CH(\mathbb{F}) \xrightarrow{H_\bullet} \mathrm{VECT}_{\mathbb{F}}}$$

is a composite of two functors.

b) Every persistence module is a functor of the form

$$\boxed{(\mathbb{N}, \le) \longrightarrow \mathrm{VECT}_{\mathbb{F}}}$$
discrete
$\bullet \to \bullet \to \bullet \to \bullet$

or

$$\boxed{(\mathbb{R}, \le) \longrightarrow \mathrm{VECT}_{\mathbb{F}}}$$
Continuous
$\bullet \longrightarrow$

Def 5 Let K be a simplicial complex. A SHEAF over K is a functor

$$\boxed{(K, \le) \xrightarrow{\mathcal{F}} \mathrm{VECT}_{\mathbb{F}}}$$

poset of simplices, ordered by face relation

Vector spaces over $\mathbb{F}$

More elaborately, $\mathcal{F}$ assigns

- to each $\sigma \in K$ its own vector space $\mathcal{F}(\sigma)$ (STALK over σ)
- to each $\sigma \le \tau$ a linear map $\underbrace{\mathcal{F}(\sigma \le \tau)}_{\text{RESTRICTION MAP}} : \mathcal{F}(\sigma) \to \mathcal{F}(\tau)$

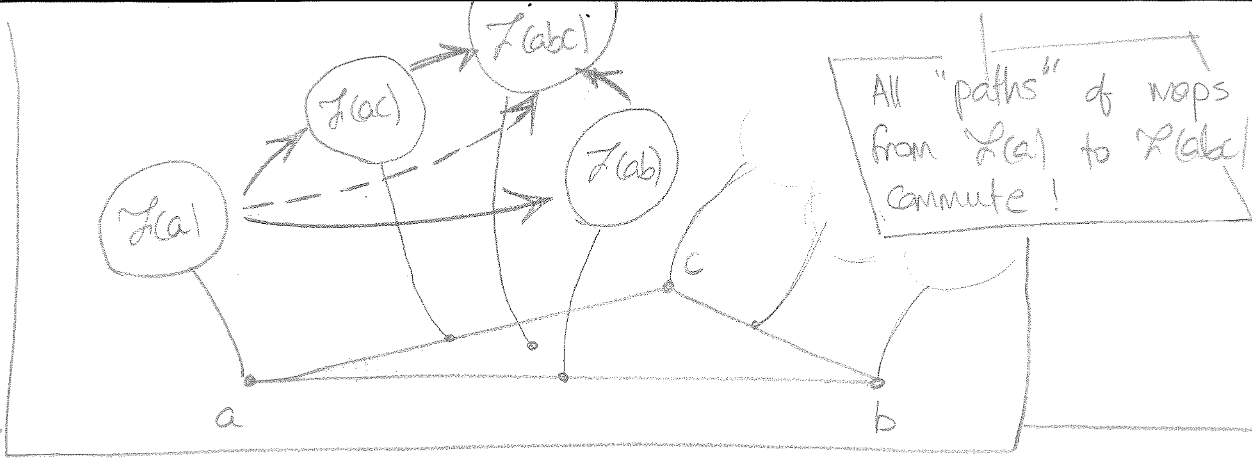
So that:

- $\mathcal{F}(\sigma \le \sigma)$ is the identity on $\mathcal{F}(\sigma)$
- the following diagram commutes: for all $\sigma \le \tau \le \gamma$:

$$\begin{tikzcd} \mathcal{F}(\sigma) \arrow[r, "\mathcal{F}(\sigma \le \tau)"] \arrow[dr, "\mathcal{F}(\sigma \le \gamma)"'] & \mathcal{F}(\tau) \arrow[d, "\mathcal{F}(\tau \le \gamma)"] \arrow[dl, phantom, "="] \\ & \mathcal{F}(\gamma) \end{tikzcd}$$

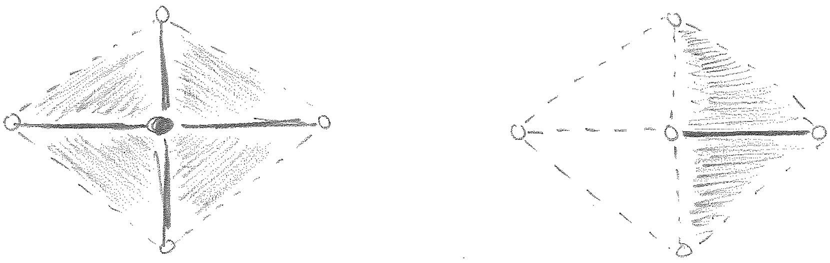
i.e., $F(\sigma \le \gamma) = F(\tau \le \gamma) \circ F(\sigma \le \tau)$ as maps $F(\sigma) \to F(\gamma)$

Fig 6



Note 7

a) Sheaves provide a principled way to study fibers of maps; for a simplicial complex L, note that the OPEN STAR of $\alpha \in L$ is the set $\{\tau \in L \mid \tau \geq \alpha\}$, so:
" $\mathrm{st}(\alpha)$



st(vertex) st(edge)

If $\alpha \leq \alpha'$, then $\mathrm{st}(\alpha) \supseteq \mathrm{st}(\alpha')$.

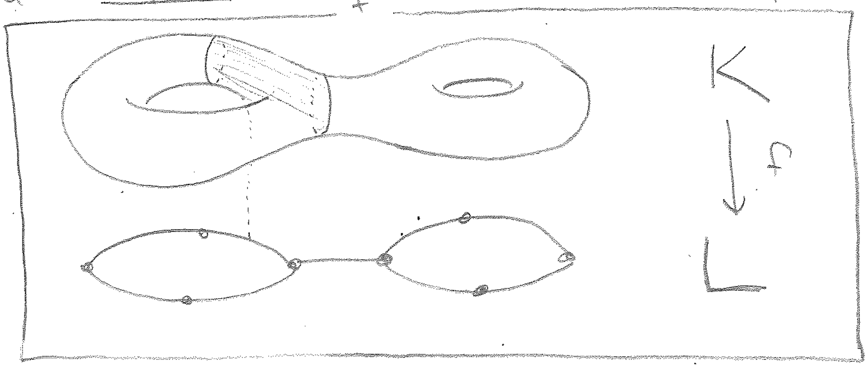
b) Given a simplicial map $f: K \to L$, the assignment

"Leray Cosheaf" of f

$$\boxed{\mathcal{L}_f^\bullet(\alpha) = H^\bullet(f^{-1}(\mathrm{st}\,\alpha); \mathbb{F})}$$

naturally produces a sheaf of $\mathbb{F}$-vector spaces over L: for $\alpha \leq \tau$, we have $\mathrm{st}(\alpha) \supseteq \mathrm{st}(\tau)$, so $f^{-1}(\mathrm{st}\,\alpha) \supseteq f^{-1}(\mathrm{st}\,\tau)$; the inclusion induces a map $\mathcal{L}_f^\bullet(\alpha) \to \mathcal{L}_f(\tau)$ going the opposite way because we are using cohomology rather than homology.

c) Can you compute $\mathcal{L}_f^\bullet$ for this example?



Q8 Okay, so what can you DO with a sheaf $\mathcal{F}$ on a simplicial complex K?

Def 9 Given $\mathcal{F}$ on K, define the vector spaces

Honestly, that $\oplus$ should be a Π. For finite K, the distinction is irrelevant

$$\boxed{C^i(K;\mathcal{F}) = \bigoplus_{\dim \sigma = i} \mathcal{F}(\sigma)}$$

direct sum of stalks assigned to all i-simplices.

This is the vector space of "chains of K with coefficients in the sheaf $\mathcal{F}$". Next, define linear maps

$$d^i_{\mathcal{F}} : C^i(K;\mathcal{F}) \longrightarrow C^{i+1}(K;\mathcal{F})$$

via the following block-action.

$$\boxed{d^i_{\mathcal{F}}\big|_{\sigma,\tau} = \langle \partial\tau, \sigma \rangle \cdot \mathcal{F}(\sigma \leq \tau)}$$

Here $\langle \partial\tau, \sigma\rangle \in \{0, \pm 1\}$ is the coefficient of σ in the simplicial boundary of τ.

Meaning, we have a block-matrix

$$d^i_{\mathcal{F}} = \begin{array}{c} \\ F(\tau) \end{array} \begin{array}{c} F(\sigma) \\ \left[\begin{array}{ccc} & \vdots & \\ \cdots & \boxed{d^i|_{\sigma,\tau}} & \cdots \\ & \vdots & \end{array}\right] \end{array}$$

[Note that $d^i|_{\sigma,\tau} \equiv 0$ when σ is NOT a face of τ].

Prop 10 The sequence

$$\boxed{C^0(K;\mathcal{F}) \xrightarrow{d^0_{\mathcal{F}}} C^1(K;\mathcal{F}) \xrightarrow{d^1_{\mathcal{F}}} \cdots}$$

is a cochain complex, i.e., $d^i_{\mathcal{F}} \circ d^{i-1}_{\mathcal{F}} \equiv 0$

Check this!!

Def 11 The COHOMOLOGY of the sheaf $\mathcal{F}$ over K is given by

$$\boxed{H^i(K;\mathcal{F}) = \ker d^i_{\mathcal{F}} \big/ \operatorname{im} d^{i-1}_{\mathcal{F}}}$$

What on earth is this "sheaf cohomology" measuring?

Eg 12 The CONSTANT SHEAF on K, written $\underline{\mathbb{F}}_K$, assigns to each σ the stalk $\mathbb{F}$ (i.e., a one-dimensional vector space) and to each $\sigma \le \tau$ the identity map $id : \mathbb{F} \to \mathbb{F}$. If you put all this information into Def 9 and Def 11, you get

$$\boxed{H^i(K, \underline{\mathbb{F}}_K) \simeq H^i(K; \mathbb{F})}$$

(sheaf) cohomology of constant sheaf

ordinary cohomology of K with $\mathbb{F}$-coefficients

(so at least we know what sheaf cohomology is measuring in this very special case...)

Def 13 a) Let $\mathscr{F}$ be a (not necessarily constant!) sheaf on a simplicial complex K, and let $L \subseteq K$ be a subcomplex. The SECTIONS of $\mathscr{F}$ over L are

$$\boxed{\Gamma(L; \mathscr{F}) = \left\{ (v_\sigma)_{\sigma \in L} \;\middle|\; v_\sigma \in \mathscr{F}(\sigma) \text{ and } \begin{array}{c} \mathscr{F}(\sigma \le \tau)[v_\sigma] = v_\tau \\ \forall\, \sigma \le \tau \end{array} \right\}}$$

i.e., each section is a choice of one element from each vector space assigned to simplices of L, which is compatible with all the restriction maps.

b) Note $\Gamma(L; \mathscr{F})$ is also an $\mathbb{F}$-vector space, we can "add" and "scale" sections term-by-term.

c) When $L = K$, the vector space $\Gamma(K; \mathscr{F})$ is called the space of GLOBAL SECTIONS of $\mathscr{F}$.

Prop 14 The zeroth sheaf cohomology group classifies global sections, i.e.,

$$\boxed{H^0(K; \mathscr{F}) \simeq \Gamma(K; \mathscr{F})}$$ (as vector spaces)

(Prove this... it's not so bad — look at $\ker d^0_{\mathscr{F}}$)

Def 15 (Redux)

A <u>COSHEAF</u> over a simplicial complex K is a functor

$$\boxed{(K, \geq) \xrightarrow{\mathcal{G}} \mathrm{VECT}_{\mathbb{F}}}$$

(opposite partial order!)

This assigns

- to each simplex σ an $\mathbb{F}$-vector space $\mathcal{G}(\sigma)$ — co-stalk
- to each face relation $\sigma \leq \tau$ a linear map

$$\boxed{\mathcal{G}(\tau) \xrightarrow{\mathcal{G}(\sigma \leq \tau)} \mathcal{G}(\sigma)}$$

co-restriction or extension map

So that $\mathcal{G}(\sigma \leq \sigma)$ is the identity, and the following diagram commutes $\forall\, \sigma \leq \tau \leq \gamma$:

$$\boxed{\begin{tikzcd} \mathcal{G}(\gamma) \arrow[r, "\mathcal{G}(\tau \leq \gamma)"] \arrow[dr, "\mathcal{G}(\sigma \leq \gamma)"'] & \mathcal{G}(\tau) \arrow[d, "\mathcal{G}(\sigma \leq \tau)"] \\ & \mathcal{G}(\sigma) \end{tikzcd}}$$

Exercise 16

a) Develop the entire story for cosheaves as was done for sheaves — note that cosheaves will have a Homology rather than a Cohomology.

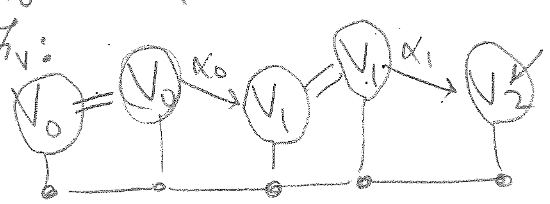
b) Note that cosheaves <u>also</u> arise naturally when we study fibers of maps — go back to NOTE 7b) and confirm that using homology rather than cohomology to define $\mathcal{L}$ will immediately produce a cosheaf.

Note 16

Every persistence module is a cosheaf over $\mathbb{R}$! Given $V_\bullet$,

$$V_0 \xrightarrow{\alpha_0} V_1 \xrightarrow{\alpha_1} V_2 \longrightarrow \cdots$$

Map to $\mathcal{I}_V$:



$\dim \Gamma([a,b]; \mathcal{I}_V)$
$= \#$ Bars in $\mathrm{Bar}(V_\bullet)$ that contain $[a,b]$.